

# Using Linear Analysis to find Eigenvalues and Eigenvectors in Power Systems

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## 1. Introduction

The behavior of a dynamic system such as a power system is dependent on the system's eigenvalues<sup>2</sup>. The response of the system is caused by the location of the eigenvalues. When linearizing a power system and plotting the eigenvalues in the complex plane, a compact description of the power system's dynamic behavior is obtained. It is worth to pay attention to analyze a power system by trying to locate its eigenvalues as they contain concentrated information of the dynamic of the system. The number of eigenvalues in a power system increases with the size of it.

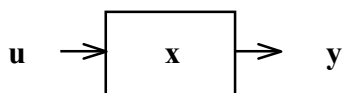
This report starts with a definition on what eigenvalues and eigenvectors in a dynamic system are and continues with an example on how they are extracted from a simple dynamic system in section 4. Eigenvalues are depicted in the complex plane and that is shown in section 5.

Section 6 points out assumptions made when performing Linear Analysis and section 7 contains information about how Linear Analysis is performed in Simpow and functions included in that tool is mentioned.

**Keywords:** Linear Analysis, eigenvalues, eigenvectors, linearization, complex plane, SIMPOW®, Sensitivity, Modal Analysis, frequency scanning, damping and data scanning.

## 2. State-Space Representation and Linearization

Every dynamic system can be described with a state vector  $\mathbf{x}$ , an input vector  $\mathbf{u}$ , and an output vector  $\mathbf{y}$ , as in figure 1. The state vector  $\mathbf{x}$  contains the  $n$  state variables of the dynamic system.



**Figure 1:** A dynamic system.

A state variable is a variable in the system that has to be time-derived through a time simulation.

If the derivatives of the state variables,  $\dot{\mathbf{x}}$ , and the output vector,  $\mathbf{y}$ , are not explicit functions of time, the dynamic

system in figure 1 can be described with the two functions  $\mathbf{f}$  and  $\mathbf{g}$ :

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u})$$

By disturbing the system in figure 1 with small deviations in both the state vector  $\mathbf{x}$  and the input vector  $\mathbf{u}$ , the linearized form of the dynamic system can be written as,

$$\Delta \dot{\mathbf{x}} = \mathbf{A} \Delta \mathbf{x} + \mathbf{B} \Delta \mathbf{u} \quad (1)$$

$$\Delta \mathbf{y} = \mathbf{C} \Delta \mathbf{x} + \mathbf{D} \Delta \mathbf{u} \quad (2)$$

The matrix  $\mathbf{A}$  in equation (1) is the state matrix of size  $[n \times n]$  and the poles of the dynamic system are the roots of the equation,

$$\det(s\mathbf{I} - \mathbf{A}) = 0 \quad (3)$$

## 3. Eigenvalues and Eigenvectors

The values of  $s$  which satisfy (3) are known as *eigenvalues* of matrix  $\mathbf{A}$ , and equation (3) is referred to as the *characteristic equation* of matrix  $\mathbf{A}$ , see [1], page 707.

The number of eigenvalues are always the same as the dimension of matrix  $\mathbf{A}$ . Some of them can be multiple. The eigenvalues can be complex and therefore it is useful to depict them in the complex plane, see section 5.

For every eigenvalue there exist two eigenvectors, *right* and *left eigenvector*.

For any eigenvalue  $\lambda_i$ , the column vector of dimension  $n$ ,  $\Phi_i$ , which satisfies equation (4) is called the *right eigenvector* of  $\mathbf{A}$ , see [1], page 707.

$$\mathbf{A} \Phi_i = \lambda_i \Phi_i \quad (4)$$

The right eigenvector measures the activity of the state variables of mode<sup>3</sup>  $i$ , see [4]. It shows how observable an eigenvalue is among the state variables.

The right eigenvector can be calculated in Simpow.

<sup>2</sup> The prefix *eigen* is borrowed from German.

<sup>3</sup> Modes and eigenvalues are similar.

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For any eigenvalue  $\lambda_i$ , the column vector of dimension  $n$ ,  $\Psi_i$ , which satisfies equation (5) is called the *left eigenvector* of  $\mathbf{A}$ .

$$\Psi_i^T \mathbf{A} = \lambda_i \Psi_i^T \quad (5)$$

The *left eigenvector* weighs the contribution of the activity of the state variables to mode  $i$ , see [4]. It shows how able a state variable is to influence an eigenvalue.

The left eigenvector cannot be calculated in Simpow.

The eigenvectors can be normalized so that the product of them in each case is,

$$\Psi_i^T \Phi_i = 1 \quad (6)$$

Note that the product of the left and right eigenvectors that belong to different eigenvalues is 0,

$$\Psi_j^T \Phi_i = 0 \quad \text{for } i \neq j$$

In the example of a linearization in section 4, both eigenvectors are calculated.

#### 4. An example of a Linearization

Below follows one example on how a simple regulator is linearized. The regulator has four possible operating points. One operating point wherein none of the limited block functions are limited, two operating points wherein one of them are limited and one where both of them are limited.

The regulator is here linearized in the operating point wherein none of the limiters are limited.

##### 4.1 Linearization

In the following linearization of the regulator in figure 2, both limiters are unlimited.

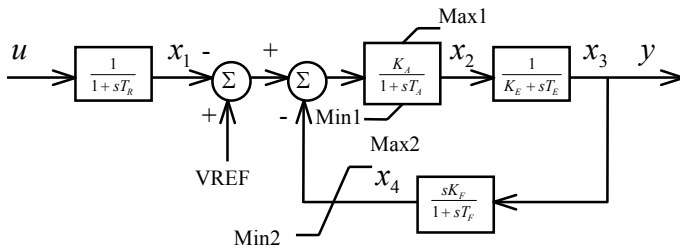


Figure 2: The regulator in unlimited operation.

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \\ \Delta \dot{x}_3 \\ \Delta \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_R} & 0 & 0 & 0 \\ -\frac{K_A}{T_A} & -\frac{1}{T_A} & 0 & -\frac{K_A}{T_A} \\ 0 & \frac{1}{T_E} & -\frac{K_E}{T_E} & 0 \\ 0 & \frac{K_F}{T_E T_F} & -\frac{K_E K_F}{T_E T_F} & -\frac{1}{T_F} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{T_R} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta u \quad (7)$$

$$\Delta y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \Delta u \quad (8)$$

The system contains of four state variables, i.e. four variables that has to be time-derived.

When applying expression (1) and (2) on the regulator the linear form in equation (7) and (8) appears.

By assuming the following values of the constants in the regulator,

$$\begin{aligned} K_A &= 200.0 & K_E &= 1.0 & K_F &= 0.05 \\ T_A &= 0.89 & T_E &= 0.80 & T_F &= 0.60 \\ T_R &= 0.02 \end{aligned}$$

equation (7), including the  $\mathbf{A}$ -matrix, is;

$$\begin{bmatrix} \Delta \dot{x}_1 \\ \Delta \dot{x}_2 \\ \Delta \dot{x}_3 \\ \Delta \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -50 & 0 & 0 & 0 \\ -224.72 & -1.12 & 0 & -224.72 \\ 0 & 1.25 & -1.25 & 0 \\ 0 & 0.10 & -0.10 & -1.67 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta u \quad (9)$$

By using equation (3), the eigenvalues of  $\mathbf{A}$  can be calculated.

The eigenvalues of the system are:

$$\begin{aligned} \lambda_1 &= -50.0 \quad (= -1/T_R) \\ \lambda_2 &= -0.0823 \\ \lambda_{3,4} &= -1.9790 \pm j0.7882 \end{aligned}$$

The real part of the eigenvalues are given in 1/sec and the imaginary part of eigenvalues  $\lambda_{3,4}$  are given in Hz.

Together with each eigenvalues, the right and left eigenvectors are calculated, see below. The eigenvectors are normalized so that the product of the left and right eigenvectors for each eigenvalue is equal to 1, see equation (6). The right eigenvectors are:

$$\begin{aligned} \Phi_1 &= \begin{bmatrix} 1 \\ 4.66 \\ -0.12 \\ -0.01 \end{bmatrix} & \Phi_2 &= \begin{bmatrix} 0 \\ -9.33 \\ -9.99 \\ 0.04 \end{bmatrix} \\ \Phi_3 &= \begin{bmatrix} 0.00 \angle 0^\circ \\ 21.45 \angle 77^\circ \\ 5.35 \angle -21^\circ \\ 0.48 \angle -3^\circ \end{bmatrix} & \Phi_4 &= \begin{bmatrix} 0.00 \angle 0^\circ \\ 21.45 \angle -77^\circ \\ 5.35 \angle 21^\circ \\ 0.48 \angle 3^\circ \end{bmatrix} \end{aligned}$$

Below, the left eigenvectors for each eigenvalue are shown,

$$\Psi_1^T = [1 \quad 0.00 \quad 0.00 \quad 0.00]$$

$$\Psi_2^T = [0.03 \quad -0.01 \quad -0.09 \quad 1]$$

$$\Psi_3^T = [0.10 \angle 88^\circ \quad 0.02 \angle -86^\circ \quad 0.02 \angle 82^\circ \quad 1 \angle 0^\circ]$$

$$\Psi_4^T = [0.10 \angle -88^\circ \quad 0.02 \angle 86^\circ \quad 0.02 \angle -82^\circ \quad 1 \angle 0^\circ]$$

### 5. Eigenvalues depicted in the complex plane

Eigenvalues are complex and the real and imaginary parts of it influence a dynamic situation. This section gives an explanation on what physical meaning the real and imaginary parts of an eigenvalue has.

Expression (10) an eigenvalue shows that the imaginary part,  $\omega$ , oscillates and the real part,  $\sigma$ , damps the oscillation (if it is negative).

$$\begin{aligned} \operatorname{Re}[e^{\lambda t}] &= \operatorname{Re}[e^{(\sigma + j\omega)t}] = \operatorname{Re}[e^{\sigma t} e^{j\omega t}] = \\ &= \operatorname{Re}[e^{\sigma t} (\cos \omega t + j \sin \omega t)] = e^{\sigma t} \cos \omega t \end{aligned} \quad (10)$$

Simpow defines eigenvalues as:  $\lambda = \sigma + j\omega$ , where  $\omega$  is given in Hertz (not in radians/second).  $\sigma$  is the damping constant and  $\omega$  is the angle speed. A damped eigenvalue has  $\sigma < 0$ .

In expression (10) we can see that if  $\sigma < 0$ , the expression will asymptotically be equal to zero as time increases. The more negative  $\sigma$ , the better the eigenvalue is damped, i.e. the faster the expression decreases to zero.

It is common to place eigenvalues in the complex plane. If an eigenvalue is damped,  $\sigma < 0$ , then it is located in the left-half plane, LHP. The left-half plane is the open left half of the complex plane, excluding the imaginary axis ( $j\omega$ -axis), see [5] page 14. Eigenvalues in the left-half plane are called stable eigenvalues.

Eigenvalues can be categorized in two types; real-valued and non-real-valued. Real-valued eigenvalues (see section 5.2) influence a dynamic simulation with pure damping.

Non-real-valued eigenvalues (see section 5.1) exist in complex pairs (with inversed imaginary parts) and influence a dynamic simulation with both damping and oscillation.

The following graphs show how these two types of eigenvalues influences the time plane.

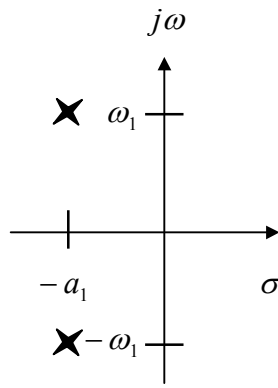
#### 5.1 A complex pair of eigenvalues

The two graphs in figure 3 shows a complex pair of eigenvalues and how they influence the time plane.

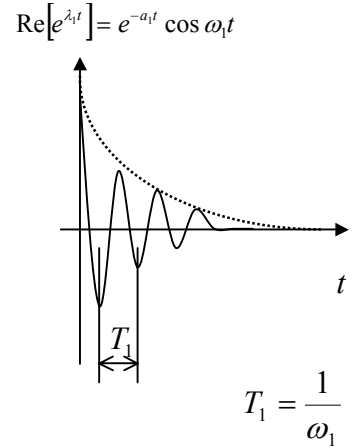
$$\lambda_{1,2} = \sigma_1 \pm j\omega_1 = -a_1 \pm j\omega_1$$

$$\operatorname{Re}[e^{\lambda_1 t}] = \operatorname{Re}[e^{\sigma_1 t} (\cos \omega_1 t + j \sin \omega_1 t)] = e^{-a_1 t} \cos \omega_1 t$$

#### Complex plane



#### Time plane



**Figure 3:** The complex plane and the time plane for a complex pair of eigenvalues.

The dashed line in the right figure describes an asymptotic damping that is a result of the exponential factor  $e^{-a_1 t}$ .

The information in figure 3 tells us that a dynamic system containing eigenvalues such as  $\lambda_1$  and  $\lambda_2$  reacts throughout a dynamic simulation with a damped ringing phenomenon as depicted in the time plane in figure 3. The frequency of the oscillation is  $\omega_1$  [Hz] and the period of the oscillation is  $T_1$ .

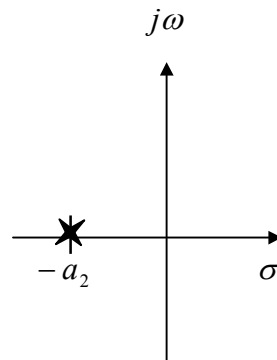
#### 5.2 A real-valued eigenvalue

The two graphs in figure 4 shows a negative real-valued eigenvalue and how it influences the time plane.

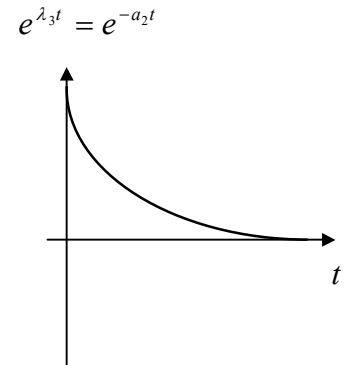
$$\lambda_3 = \sigma_2 = -a_2$$

$$e^{\lambda_3 t} = e^{\sigma_2 t} = e^{-a_2 t}$$

#### Complex plane



#### Time plane



**Figure 4:** The complex plane and the time plane for a real-valued eigenvalue.

The information in figure 4 tells us that a dynamic system containing an eigenvalue such as  $\lambda_3$  reacts throughout a dynamic simulation with a damping phenomenon as depicted in the time plane in figure 4.

## 6. Assumptions made in Linear Analysis

In Linear analysis the power system is linearized in the current working point. As a result of this, non-linear components is fixed in one mode during the analysis and their discrete events is excluded in Linear Analysis. Because of this, different modes of the power system generate different eigenvalues when Linear Analysis is done in different points of time. Linear Analysis made on a power system in Transta mode in Simpow makes it possible to observe electro-mechanical oscillations. These oscillations are about a few Hertz. With a power system represented in Masta mode in Simpow it is possible to observe for example sub-synchronous resonance and harmonics.

## 7. Linear Analysis functions in Simpow

The following activities are implemented in Simpow's tool Linear Analysis.

- i) Calculation of Eigenvalues
- ii) Frequency Scanning
- iii) Damping Scanning
- iv) Data Scanning
- v) Eigenvector Computation
- vi) Response Vector Computation
- vii) Modal Analysis
- viii) Sensitivity
- ix) Harmonics Distribution

### i) Calculation of Eigenvalues

Analyzes the power system and calculates the eigenvalues that the system contains. Eigenvalues are at first calculated with the QR-method and when one eigenvalue has been chosen, another, more detailed method is used called inverse iteration method.

### ii) Frequency Scanning

A frequency-variant disturbance is injected somewhere in the power system, for example a positive-sequence current in a node. Among others, voltages can be observed at nodes in the system. This is the same as scanning along the  $j\omega$ -axis (y-axis) in the complex plane. Frequency scanning shows the results graphically in the frequency plane.

Frequency Scanning parallel with the  $j\omega$ -axis (y-axis) can be performed by defining a value of  $\sigma$ .

### iii) Damping Scanning

A damping-variant disturbance is applied somewhere in the power system and for example voltage changes can be observed at nodes in the system. This is the same as scanning along the  $\sigma$ -axis (x-axis) in the complex plane.

Damping Scanning parallel with the  $\sigma$ -axis (x-axis) can be performed by defining a value of  $\omega$ .

### iv) Data Scanning

The influence on an eigenvalue when changing one parameter can be observed with Data Scanning. Data Scanning complements the tool Sensitivity, see viii) below.

Here we can see what impact a certain parameter has on an eigenvalue, how the eigenvalue moves in the complex plane when a parameter is changed in a specified interval.

With Data Scanning it is possible to further analyze what can be seen with Sensitivity, see viii) below.

### v) Eigenvector Computation

Computation of the right eigenvector,  $\Phi_i$ , corresponding to an eigenvalue,  $\lambda_i$ , see section 3 above.

### vi) Response Vector Computation

Shows how each variable in the power system are influenced by a disturbance in the complex plane, for example a disturbance describing an eigenvalue.

### vii) Modal Analysis

With Modal Analysis it is possible to see how different synchronous generators feeds a selected eigenvalue.

### viii) Sensitivity

With Sensitivity it is possible to see how different system components and their parameters influence eigenvalues.

### ix) Harmonics Distribution

The system is modeled frequency-dependent with only Masta-models. This allows a good view of how harmonics are distributed in a power system. Rotating machines in the network must be modeled as infinite buses.

## 8. References

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